REVIEW ARTICLE

Fuzzy Regular Topological Space in Quasi-coincidence Sense

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ABSTRACT

In this paper, we introduce two notions of regular property in fuzzy topological spaces by using quasi-coincidence sense and we establish relationship among them. We also show that all these notations satisfy hereditary property. It is observed that one-one, onto, fuzzy open, fuzzy closed and fuzzy continuous mappings are satisfied by authors' concepts.

Keywords: ***Fuzzy Topological Space, Fuzzy Regular Topological Space, Mappings, Quasi-coincidence.***

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1. INTRODUCTION

The concept of fuzzy set is first presented in 1965 [1]. By using this concept, Chang CL [2] defined fuzzy topological spaces in 1968. Since then, extensive work on fuzzy topological spaces has been carried out by many researchers like Gouguen [3], Wong [4], Lowen [5], Warren [6], Hutton [7] and others. Separation axioms are important parts in fuzzy topological spaces. Many works [8, 9, 10, 11, 12, 13] on separation axioms have been done by researchers. Among those axioms, fuzzy regular type is one and it has been already introduced in the literature. There are many articles on fuzzy regular topological space which are created by many authors like P. Wuyts and R. Lowen [14], Ali DM [8], Guler AC and kale G [15] and many others.

The purpose of this paper is to further contribute to the development of fuzzy topological spaces especially on fuzzy regular topological spaces. In the present paper, fuzzy regular topological space is defined by using quasi-coincidence sense and relations. In the next section of this paper, it is showed that the hereditary and order preserving properties hold on the new concepts.

2. BASIC NOTIONS AND PRELIMINARY RESULTS

In this section, we recall some concepts occurring in the papers [8] which will be needed in the sequel. In this paper "⇒" indicates "implies", X and Y are always denoted as non empty sets and *I*= [0,1]. The class of all fuzzy sets on a non empty set *X* is denoted by I^X and fuzzy sets on *X* are denoted as u, v, w, p etc.

Definition 2.1 [1]

A function u from *X* into the unit interval *I* is called a fuzzy set in *X*. For every $x \in X$, $u(x) \in I$ is called the grade of membership of x in u. Some authors say that u is a fuzzy subset of X instead of saying that u is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X.

Definition 2.2 [16]

A fuzzy set u in X is called a fuzzy singleton if and only if $u(x) = r$, $0 < r \leq 1$, for a certain $x \in X$ *and* $u(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be

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denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if $r \leq u(x)$.

Definition 2.3 [17]

A fuzzy singleton x_r is said to be quasi-coincidence with u, denoted by $x_r q u$ if and only if $u(x) + r > 1$. If x_r is not quasi-coincidence with u, we write $x_r \bar{q} u$ and defined as $u(x) + r \leq 1$.

Definition 2.4 [2]

Let f be a mapping from a set X into a set Y and u be a fuzzy subset of X. Then f and u induce a fuzzy subset v of Y defined by-

$$v(y) = sup\{u(x)\}$$

if $x \in f^{-1}[\{y\}] \neq \varphi, x \in X \ = 0$ otherwise.

Definition 2.4 [2]

Let f be a mapping from a set X into a set Y and v be a fuzzy subset of Y . Then the inverse of v written as $f^{-1}(v)$ is a fuzzy subset of X defined by $f^{-1}(v)(x) = v(f(x))$, for x ∈ X.

Definition 2.5 [2]

Let I = [0, 1], X be a non empty set and I^X be the collection of all mappings from X into I, i.e. the class of all fuzzy sets in X. A fuzzy topology on X is defined as a family t of members of I^X, satisfying the following conditions.

i. $1, 0 \in t$,

ii. If $u \in t$ for each $i \in \Lambda$, then $\bigcap_{i \in \Lambda} u_i \in t$, where Λ is an index set.

iii. If u, $v \in t$ then $u \cap v \in t$.

The pair (X, t) is called a fuzzy topological space (in short fts) and members of t are called t- open fuzzy sets. A fuzzy set v is called a t-closed fuzzy set if $1 - v \in t$.

Definition 2.6 [18]

The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

Definition 2.7 [19]

The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set u in (X, t), $f(u)$ is open fuzzy set in (Y, s).

Definition 2.8 [18]

The function $f : (X, t) \to (Y, s)$ is called fuzzy closed if and only if for every closed fuzzy set u in (X, t), $f(u)$ is closed fuzzy set in (Y, s).

Theorem 2.1.1 [20]

A bijective mapping from an fts (X, t) to an fts (Y, s) preserves the value of a fuzzy singleton (fuzzy point).
Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

3. The Main Results

In this section, we discuss about our notions and findings. Some well-known properties are discussed here by using our concepts.

Definition 3.1

A fuzzy topological space (X, t) is called

(a) $FR(i)$ if $x_r \notin w$, w is t-closed fuzzy set, there exist $u, v \in t$ such that $x_r \in u, w \subseteq v$ and $u \bar{q} v$.

(b) $FR(ii)$ if $x_r \notin w$, w is t-closed fuzzy set, there exist $u, v \in t$ such that $x_r \in u$, $w \subseteq v$ and $u \cap v = 0$.

Theorem 3.1.1

If (X, t) is a fuzzy topological space, then we will get the following implication

$$FR(ii) \Rightarrow FR(i).$$

But the converse is not true.

Proof: $\boldsymbol{FR(ii) \Rightarrow FR(i)}$: Let (X, t) be a fuzzy topological space and $(X,\ t)$ is $FR(ii)$. We have to prove that $(X,\ t)$ is $FR(i)$. To prove $(X,\ t)$ is $FR(i)$, it is only needed to prove that $u\bar{q}v$.

Now, $u \cap v = 0$

$$\Rightarrow (u \cap v)(x) = 0$$

$$\Rightarrow \min\left(u(x), v(x)\right) = 0$$

$$\Rightarrow u(x) + v(x) \leq 1$$

$$\Rightarrow u\bar{q}v.$$

Hence, $(X,\ t)$ is $FR(i)$. To show $FR(i) \nRightarrow FR(ii)$, we give a counter example.

Counter example: Let $X = \{x,\ y\}$ and $u, v \in I^X$ be given by $u(x) = 0.8,\ u(y) = 0.1,$

$v(y) = 0.8, v(x) = 0.1.$

Let us consider the fuzzy topology t on X generated by $\{0,\ u,\ v, 1\}$. For $0.2 < r \leq 1$,

$$u(x) + v(x) \leq 1$$

$$\Rightarrow u\bar{q}v\cdot$$

But $$\min\left(u(x), v(x)\right) = 0.1 \neq 0$$

$$\Rightarrow (u \cap v)(x) \neq 0$$

$$\Rightarrow (u \cap v) \neq 0.$$

Also, $u(y) + v(y) \leq 1 \Rightarrow u\bar{q}v.$

Hence it is clear that $(X,\ t)$ is $FR(i)$.

But $$\min\left(u(y), v(y)\right) = 0.1 \neq 0$$

$$\Rightarrow (u \cap v)(y) \neq 0$$

$$\Rightarrow (u \cap v) \neq 0$$

Hence it is clear that $(X,\ t)$ is not $FR(ii)$. These complete the proof of the implications.

Now we shall show that our notions satisfy the hereditary property.

Theorem 3.1.2

Let $(X,\ t)$ be a fuzzy topological space, $A \subseteq X, t_A = \{u/A : u \in t\}$, then

(a) $(X,\ t)$ is $FR(i) \Rightarrow (A, t_A)$ is $FR(i)$ and

(b) $(X,\ t)$ is $FR(ii) \Rightarrow (A, t_A)$ is $FR(ii)$.

Proof (a): Let (X, t) is $FR(i)$. It is required to prove that $(A,\ t_A)$ is $FR(i)$. Let w be a t_A-closed fuzzy set and x_r be a fuzzy singleton in A such that $w(x) < r$. This implies that $w^c \in t_A$ and $w^c(x) > 1 - r$. So there exists an $u \in t$ such that $u/A = w^c$ and clearly u^c is closed in t. Now, $u^c(x) = (u/A)^c(x) = w(x) < r$ since $x \in A$.

Since (X, t) is $FR(i)$, then there exists $p, v \in t$ such that $x_r \in p, u^c \subseteq v$ and $p\bar{q}v$. Since $p, v \in t$, then $p/A, v/A \in t_A$. Now, we have

$x_r \in p/A$, $(u/A)^c \subseteq v/A$ and $(p/A)\bar{q}(v/A)$.

So $x_r \in p/A$, $w \subseteq v/A$ and $(p/A)\bar{q}(v/A)$. Therefore $(A,\ t_A)$ is $FR(i)$.

Proof of (b) is similar to proof of (a).

Now, we shall show that our notions satisfy the order preserving property.

Theorem 3.1.3

If (X, t) and (Y, s) are two fuzzy topological spaces and $f: X \rightarrow Y$ is one-one, onto, fuzzy continuous and fuzzy open mapping. Then

$(a)(X,\ t)$ is $FR(i) \Rightarrow (Y,\ s)$ is $FR(i)$

$(b)(X,\ t)$ is $FR(ii) \Rightarrow (Y,\ s)$ is $FR(ii)$

Proof (a): Let $(X,\ t)$ be a fuzzy topological space and $(X,\ t)$ is $FR(i)$. We have to prove that $(Y,\ s)$ is $FR(i)$. Let w be s-closed and y_r be a singleton in Y with $w(y) < r$.

Then $f^{-1}(w) \in t^c$ as f is fuzzy continuous. Since f is bijective, then for $y \in Y$ there exits $x \in X$ such that $f(x) = y$. Now,

$f^{-1}(w)(x) = w(f(x)) = w(y) < r$. Since (X,t) is $FR(i)$, then there exist $u, v \in t$ such that $x_r \in u, f^{-1}(w) \subseteq v$ and $u\bar{q}v$.

Now, we see that

$$f(u)(y) = \{\sup u(x) : f(x) = y\} < r.$$

So, $y_r \in f(u)$.

Also, $u\bar{q}v$ implies that $u(x) + v(x) \leq 1$ for all $x \in X$.

Now for all $f(x) \in Y$, we have $f(u)(f(x)) + f(v)(f(x)) = u(x) + v(x) \leq 1$ as f is bijective. So, $f(u)\bar{q}f(v)$.

Again, since $f^{-1}(w) \subseteq v$, then $w \subseteq f(v)$. It is clear that $f(u) \in s_1$, $f(v) \in t_1$ as f is FP-open. So $f(u), f(v) \in s$ such that $y_r \in f(u)$, $w \subseteq f(v)$ and $f(u)\bar{q}f(v)$. Hence (Y,s) is $FR(i)$. Proof of (b) is similar to proof of (a).

Theorem 3.1.4

If (X, t) and (Y, s) are two fuzzy topological spaces and f: X →Y is one-one, onto, fuzzy closed and fuzzy continuous mapping. Then,

(a) $(Y,\ s)\ is\ FR(i) \Rightarrow (X,\ t)\ is\ FR(i)$

(b) $(Y,\ s)\ is\ FR(ii) \Rightarrow (X,\ t)\ is\ FR(ii)$.

Proof (a): Let the fuzzy topological space (Y,s) is $FR(i)$. It is required to prove that (X,t) is $FR(i)$. Let w be t-closed and x_r be a singleton in X with $w(x) < r$.

Then $f(w) \in s^c$ as f is fuzzy closed and let $f(x) = y$. Now we have

$f(w)(y) = \{\sup w(x) : f(x) = y\} < r$ since f is one-one.

Since (Y,s) is $FR(i)$, then there exist $u, v \in s$ such that

$y_r \in u, f(w) \subseteq v$ and $u\bar{q}v$.

Now, we see that

$$f^{-1}(u)(x) = u(f(x)) = u(y) \leq r.$$

So $x_r \in f^{-1}(u)$.

Also, it is clear that $f(w) \subseteq v$ implies that $w \subseteq f^{-1}(v)$ as f is bijective.

Again $u\bar{q}v$ implies that $u(y) + v(y) \leq 1$ for all $y \in Y$.

Now for all $x \in X$, we have

$$f^{-1}(u)(x) + f^{-1}(v)(x) = u(f(x)) + v(f(x)) = u(y) + v(y) \leq 1.$$

So $f^{-1}(u)\bar{q}f^{-1}(v)$.

Since f is fuzzy continuous, then $f^{-1}(u),\ f^{-1}(v) \in t$. It follows that $f^{-1}(u), f^{-1}(u) \in t$ such that $x_r \in f^{-1}(u)$, $w \subseteq f^{-1}(v)$ and $f^{-1}(u)\bar{q}f^{-1}(v)$. Hence (X,t) is $FR(i)$.

Similarly, one can prove (b).

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